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Nonexistence of Massless Zero-Spin Particles in Relativistic Quantum Mechanics

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Abstract

The Klein–Gordon equation is cast, using its two-components version, into a form which exactly parallels the Dirac equation and which is used to discuss the Klein–Gordon analogs of the unitary and nonunitary transformations of physical interest appearing in the latter. In particular, it is found that massless zero-spin particles do not exist within the framework of this theory.

1. *Introduction*

Massless zero-spin particles, i.e., Goldstone bosons (Goldstone, 1961) have been most important in field theoretical research during the past few years. In this paper we discuss a closely related topic, namely, the description of a massless zero-spin particle within the framework of relativistic quantum mechanics.

As is well known, a quantum mechanical description of spin-1/2 and spin-1 massless particles can easily be obtained from the Dirac and Kemmer equations, respectively, by taking the ultrarelativistic limit (i.e., the limit $|\mathbf{p}| \gg m$) in a suitable manner in each case (in fact, by just setting $m = 0$ in the Dirac case). A unified way of doing this, however, is naturally provided by the Cini–Touschek (1958) transformation, as recently stressed by us (Saavedra, 1973).

In the present paper we want to extend this technique to the Klein–Gordon equation†,

$$(\Box + m^2)\psi(x) = 0 \tag{1.1}$$

For this purpose, we use the two-component version of Eq. (1.1), in which the time derivatives appear in the first order only. This is a formalism essentially due to Taketani and Sakata (1940) and Case (1954), although it has become

† We use the metric $g_{00} = -g_{kk} = +1$ and set $\hbar = c = 1$.

customary to refer to it as the "Feshbach and Villars (1958) formalism" in the literature. The resulting wave equation is the Hamiltonian equation,

$$i\partial_t \Phi = H_{KG}\Phi \tag{1.2}$$

with the operator H_{KG} given by

$$H_{KG} = -(\nabla^2/2m)(\sigma_3 + i\sigma_2) + m\sigma_3 \tag{1.3}$$

the σ's being Pauli matrices.

Now, introducing the angular momentum operators

$$\boldsymbol{\tau} \equiv \tfrac{1}{2}\boldsymbol{\sigma}, \qquad \boldsymbol{\tau} \times \boldsymbol{\tau} = i\boldsymbol{\tau} \tag{1.4}$$

and working in momentum space, the Hamiltonian (1.3) can be rewritten in the form

$$H_{KG} = 2(i\mathbf{p}^2/2m)\tau_2 + 2(\mathbf{p}^2/2m + m)\tau_3 \tag{1.5}$$

which can be compared with the Dirac Hamiltonian,

$$H_D = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m \tag{1.6}$$

when written in terms of the angular momentum operators $\mathbf{J}$,

$$J_1 \equiv \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2|\mathbf{p}|}, \; J_2 \equiv -i\beta\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2|\mathbf{p}|}, \; J_3 \equiv \frac{\beta}{2}, \; \mathbf{J}\times\mathbf{J} = i\mathbf{J} \tag{1.7}$$

which we have introduced previously (Saavedra, 1970, 1973), namely,

$$H_D = 2|\mathbf{p}|J_1 + 2mJ_3 \tag{1.8}$$

The analogy in the structure of the Hamiltonians (1.5) and (1.8) is further stressed by the introduction of the notation

$$P \equiv i\frac{\mathbf{p}^2}{2m}, \qquad M \equiv \frac{E^2 + m^2}{2m}, \qquad E = {}_{+}\sqrt{\mathbf{p}^2 + m^2} \tag{1.9}$$

in terms of which Eq. (1.5) reads

$$H_{KG} = 2P\tau_2 + 2M\tau_3 \tag{1.10}$$

and by remarking that the following relation is satisfied [Eq. (1.9)]

$$E^2 = \mathbf{p}^2 + m^2 = P^2 + M^2 \tag{1.11}$$

It is the purpose of this paper to exploit this remarkable analogy to obtain for the Klein–Gordon case the transformations corresponding to the unitary [e.g., Foldy and Wouthuysen (1950)] and nonunitary (e.g., "Lorentz") transformations of physical interest appearing in the Dirac case. Having done this, we shall show that a nontrivial difference between the two theories arises because of the correspondence $J_1 \to \tau_2$ (instead of τ_1) exhibited by Eqs. (1.8) and (1.10). In particular, by exploring the extreme-relativistic limit of the Klein–Gordon equation, we shall show that massless zero-spin particles do not exist in the framework of relativistic quantum mechanics, unless their *charge density*, $\rho(\mathrm{x})$, also vanished identically. It is clear, however, that the latter objects would have no meaning within this theory, as they would have no measurable properties, because the theory has no further degrees of freedom allowing for the description of additional quantum numbers.

2. *Résumé of the Taketani-Sakata-Case Formalism*

As it is essential for our discussion, we shall quote here the results of the Taketani-Sakata-Case formalism we shall be using in what follows.

The wave function $\Phi(x)$ [Eq. (1.3)] is written as

$$\Phi(x) = \begin{pmatrix} \phi_0(x) \\ \chi_0(x) \end{pmatrix} \tag{2.1}$$

where

$$\begin{aligned} \phi_0 &= \tfrac{1}{2}[\psi + (i/m)\partial_t\psi] \\ \chi_0 &= \tfrac{1}{2}[\psi - (i/m)\partial_t\psi] \end{aligned} \tag{2.2}$$

with ψ satisfying Eq. (1.1). The Klein-Gordon density

$$\rho(x) = (i/2m)\psi^*(x)\overleftrightarrow{\partial_t}\psi(x) \tag{2.3}$$

reads then

$$\rho(x) = |\phi_0|^2 - |\chi_0|^2 = \Phi^+(x)\sigma_3\Phi(x) \tag{2.4}$$

which suggests the convenience of introducing the following definition for the scalar product in the space of the vectors $\Phi(x)$:

$$\langle\Phi_1(x)|\Phi_2(x)\rangle = : (\Phi_1(x), \sigma_3\Phi_2(x)) \equiv \int d^3x\Phi_1^+(x)\sigma_3\Phi_2(x) \tag{2.5}$$

In particular, the norm of the vector Φ is now

$$N(\Phi) = \int d^3x(|\phi_0|^2 - |\chi_0|^2) \tag{2.6}$$

To simplify the notation we shall omit the integral sign in what follows, i.e., we shall write simply

$$N(\Phi) = |\phi_0|^2 - |\chi_0|^2 = \Phi^+\sigma_3\Phi \tag{2.7}$$

implying, of course, Eq. (2.6).

The mean value of an operator Ω is defined as

$$\langle\Omega\rangle = : (\Phi, \sigma_3\Omega\Phi)/N(\Phi) \tag{2.8}$$

and the requirement that it must be a real number implies that Ω must be a *pseudo-Hermitian* operator, satisfying

$$\Omega = \sigma_3\Omega^+\sigma_3 \tag{2.9}$$

The Hamiltonian (1.3) is clearly a Hermitian operator in this formalism (we shall omit the prefix "pseudo" in what follows).

In the same way, we shall say that the operator U transforming

$$\Phi \to \Phi' = U\Phi$$

is (pseudo) *unitary* if the transformation preserves the norm $[N(\Phi) = N(\Phi')]$, i.e., if the operator U satisfies the relation

$$U^{-1} = \sigma_3 U^+ \sigma_3 \tag{2.10}$$

As is well known, the Klein-Gordon amplitude $\psi(x)$ is not a probability amplitude because the corresponding density [Eq. (2.3)] can assume negative values; in the Taketani-Sakata-Case formalism, on the other hand, $\rho(x)$ [Eq. (2.4)] can be understood as a *charge density* and therefore can assume negative values without running into contradictions. To see this it is only necessary to introduce the electromagnetic field in the usual way

$$P_\mu \to P_\mu - eA_\mu$$

and verify by direct calculation that if the function $\Phi(x)$ is a solution of the resulting wave equation for a given sign of the electric charge, then the function

$$\Phi_c(x) \equiv \sigma_1 \Phi^*(x) \tag{2.11}$$

satisfies the same wave equation for the opposite sign of the charge: Φ_c is the charge-conjugated wave function to Φ. Explicitly, using Eq. (2.11) one finds

$$\rho_c = \Phi_c^+ \sigma_3 \Phi_c = |\chi_0|^2 - |\phi_0|^2 = -\rho \tag{2.12}$$

which shows that $\rho(x)$ can indeed be understood as an electrical charge density.

For later reference, we finally also quote here the explicit form of the free-particle solution of Eq. (1.2) for positive energy and normalization (+1), namely,

$$\Phi(x) = \frac{1}{2\sqrt{mE}} \begin{pmatrix} E + m \\ -E + m \end{pmatrix} e^{-ip \cdot x} \tag{2.13}$$

3. *The Foldy-Wouthuysen and Cini-Touschek Transformations*

To start with, let us briefly recall our technique (Saavedra, 1970) for obtaining these transformations in the Dirac theory. Remarking that the Hamiltonian (1.8) does not contain the operator J_2, one defines the operator

$$T = \exp(iJ_2\theta) = \exp\left(\beta \frac{\boldsymbol{\alpha} \cdot \mathbf{p}}{|\mathbf{p}|} \frac{\theta}{2}\right) \tag{3.1}$$

to "rotate" the (Dirac's) Hamiltonian equation

$$(2|\mathbf{p}|J_1 + 2mJ_3 - p_0)\psi(\mathbf{p}) = 0 \tag{3.2}$$

into an equation which does not contain either J_1 or J_3 by suitable choices of the parameter ("angle") θ. This transformation is most easily performed owing to the fact that the **J** operators are angular momentum operators and therefore satisfy the relation

$$e^{iJ_j\theta}J_k e^{-iJ_j\theta} = J_k \cos\theta - \epsilon_{jkl}J_l \sin\theta \tag{3.3}$$

If the angle θ is chosen such that

$$\tan\theta = |\mathbf{p}|/m \equiv \tan\theta_D^{\mathrm{FW}} \tag{3.4}$$

the Hamiltonian (1.8) is transformed into

$$H_D^{\mathrm{FW}} = 2J_3E = \beta E \tag{3.5}$$

This is the Foldy–Wouthuysen (1950) transformation, suitable for the discussion of the nonrelativistic limit of the Dirac equation. If, on the other hand, the angle θ is chosen such that

$$\tan\theta = -m/|\mathbf{p}| \equiv \tan\theta_D^{\mathrm{CT}} \tag{3.6}$$

the Hamiltonian (1.8) reduces to

$$H_D^{\mathrm{CT}} = 2J_1E = [(\boldsymbol{\alpha}\cdot\mathbf{p})/|\mathbf{p}|]E \tag{3.7}$$

which yields the Cini–Touschek (1958) equation, suitable to discuss the extreme-relativistic limit of the theory. Notice that the angles (3.4) and (3.6) satisfy the relation

$$\tan\theta_D^{\mathrm{FW}}\cdot\tan\theta_D^{\mathrm{CT}} = -1 \tag{3.8}$$

or, equivalently,

$$|\boldsymbol{\theta}_D^{\mathrm{FW}} - \boldsymbol{\theta}_D^{\mathrm{CT}}| = \pi/2 \tag{3.9}$$

Let us then apply this technique to the Taketani–Sakata Eq. (1.2), with the Hamiltonian given by Eq. (1.5).

The appropriate rotation operator is now

$$T = e^{i\tau_1\theta} = e^{i\sigma_1\theta/2} \tag{3.10}$$

Defining†

$$\tilde{\phi}(\mathbf{p}) = T\phi(\mathbf{p}) \tag{3.11}$$

and, using Eq. (3.3), this transformation leads to

$$\left[\sigma_2\left(i\frac{\mathbf{p}^2}{2m}\cos\theta + \frac{E^2+m^2}{2m}\sin\theta\right) + \sigma_3\left(\frac{E^2+m^2}{2m}\cos\theta - i\frac{\mathbf{p}^2}{2m}\sin\theta\right) - p_0\right]\tilde{\phi}(\mathbf{p}) = 0 \tag{3.12}$$

† We work with plane wave solutions, $\Phi(x) = \phi(\mathbf{p})e^{-ip\cdot x}$, and the wave equation we consider (in momentum space) is $H\phi(\mathbf{p}) = p_0\phi(\mathbf{p})$.

If we choose to eliminate σ_2, the corresponding angle is given by

$$\tan\theta = -i\,\mathbf{p}^2/(E^2 + m^2) \equiv \tan\theta_{\mathrm{KG}}^{\mathrm{FW}} \tag{3.13}$$

yielding the Hamiltonian

$$H_{\mathrm{KG}}^{\mathrm{FW}} = \sigma_3 E \tag{3.14}$$

which is identical in structure to H_D^{FW} [Eq. (3.5)]. This transformation was first discussed by Case (1954).

The transformation (3.10) with the angle (3.13) is a unitary transformation, i.e., it satisfies Eq. (2.10), as it is readily proved by noticing that the angle θ is imaginary; the norm is therefore conserved.

If we now choose to eliminate σ_3 in Eq. (3.12), recalling the spin-1/2 case we should expect to get the analog of the Cini–Touschek equation, i.e., an equation suitable to study the extreme relativistic limit of the Klein–Gordon equation. This turns out *not to be the case*, however.

Indeed, the term containing σ_3 in Eq. (3.12) is eliminated with the choice

$$\tan\theta = -i(E^2 + m^2)/\mathbf{p}^2 \equiv \tan\theta_{\mathrm{KG}}^{\mathrm{CT}} \tag{3.15}$$

which satisfies

$$\tan\theta_{\mathrm{KG}}^{\mathrm{CT}} \cdot \tan\theta_{\mathrm{KG}}^{\mathrm{FW}} = -1 \tag{3.16}$$

in analogy with Eq. (3.8); in this sense we can call, therefore, the corresponding transformation a Cini–Touschek transformation. This transformation, nevertheless, has no direct physical meaning. For, defining $\theta_{\mathrm{KG}}^{\mathrm{CT}} \equiv i\omega$, Eq. (3.15) yields

$$|\tanh\omega| = (E^2 + m^2)/\mathbf{p}^2 > 1 \tag{3.17}$$

for all physical(real) values of m and $|\mathbf{p}|$, and therefore this transformation does not exist. It is also clear from Eq. (3.17) that this transformation can describe *tachyons*, with

$$m = im_* \tag{3.18}$$

where m_* is real. In this interpretation the resulting Hamiltonian is

$$H_{\mathrm{KG}}^{\mathrm{CT}} = -2\tau_2 E \tag{3.19}$$

which should be compared with Eq. (3.7), where the operator J_1 (and not J_2) appears. Notice also the minus sign appearing in Eq. (3.19), which does not appear in Eq. (3.7); we shall return to this point in Sec. 4.

Finally, to further stress the analogy between the Dirac and Klein–Gordon theories, we remark that the following relation holds:

$$H_{\mathrm{KG}}^{\mathrm{FW}}\cos\theta + H_{\mathrm{KG}}^{\mathrm{CT}}\sin\theta = H_{\mathrm{KG}} \tag{3.20}$$

with the angle θ given by Eq. (3.13), and that an identical equation holds for the Dirac case (Saavedra, 1967).

4. *A Geometrical Analogy*

The results (3.15)-(3.19) are most easily understood though a geometrical analogy which follows from the observation that the Foldy-Wouthuysen (F-W) and Cini-Touschek (C-T) transformations do not change the energy, i.e., they are transformations leaving invariant the quadratic form

$$E^2 = |\mathbf{p}|^2 + m^2 \tag{4.1}$$

The geometrical analog of these transformations is therefore the set of (two-dimensional) rotations in the plane $(m, |\mathbf{p}|)$, that is, the transformations

$$m' = m \cos\theta + |\mathbf{p}| \sin\theta \tag{4.2a}$$

$$|\mathbf{p}'| = -m \sin\theta + |\mathbf{p}| \cos\theta \tag{4.2b}$$

If we want to consider the nonrelativistic limit, i.e., (physically) the limit $|\mathbf{p}| \ll m$, the corresponding geometrical analog is the rotation [Eq. (4.2b)] given by

$$\tan\theta = |\mathbf{p}|/m = \tan\theta_D^{\mathrm{FW}} \tag{4.3}$$

for which $|\mathbf{p}'|$ vanishes; this is indeed the F-W angle, as given by Eq. (3.4).

We now again use the analogy between the Dirac and Klein-Gordon theories pointed out in the Introduction [Eqs. (1.8)-(1.11)] : the corresponding geometrical analog for the Klein-Gordon case is obtained from Eqs. (4.2) with the replacements

$$m \to M, \quad |\mathbf{p}| \to \mathbf{P} \tag{4.4}$$

with M and P defined by Eqs. (1.9).

The rotation (in this new plane) for which

$$P' = 0 \tag{4.5}$$

is given by

$$\tan\theta = P/M = +i\,\mathbf{p}^2/(E^2 + m^2) = -\tan\theta_{\mathrm{KG}}^{\mathrm{FW}} \tag{4.6}$$

that is, is given by Eq. (3.13) except for a minus sign; we shall show in Sec. 5 that this change in sign is indeed required by the formalism.

The physical parameters of the theory are of course m and $|\mathbf{p}|$, and not M and P, and therefore in order to give a physical interpretation to Eq. (4.6) we must first express Eq. (4.5) in terms of $|\mathbf{p}'|$, which is trivially done using Eq. (1.9) to obtain $|\mathbf{p}'| = 0$, which indeed confirms that this transformation is useful to study the nonrelativistic limit of the Klein-Gordon theory.

The extreme-relativistic limit in the Dirac case is obtained here by setting

$$m' = 0 \tag{4.7}$$

that is, by the rotation [Eq. (4.2a)]

$$\tan\theta = -m/|\mathbf{p}| = \tan\theta_D^{\mathrm{CT}} \tag{4.8}$$

where the last equality is Eq. (3.6); this is the C-T angle.

The corresponding rotation in the Klein-Gordon case yields

$$M' = 0 \tag{4.9}$$

and is given by the angle

$$\tan\theta = -M/P = +i(E^2 + m^2)/\mathbf{p}^2 = -\tan\theta_{\mathrm{KG}}^{\mathrm{CT}} \tag{4.10}$$

where again the change in sign is expected (see Sec. 5). To interpret this result we once more use Eq. (1.9) to obtain [from Eq. (4.9)]

$$p'^2 = -2m^2 \tag{4.11}$$

which says that the corresponding transformation is a *tachyon transformation* with $v = \sqrt{2}$, that is [Eq. (3.18)], $m(v) = m_*$ and $E^2 = m_*^2$. Indeed, using Eq. (4.11), Eq. (1.5) yields the wave equation

$$-\sigma_2\tilde{\tilde{\phi}} = \pm\,\tilde{\tilde{\phi}} \tag{4.12}$$

which is identical in structure to the Hamiltonian equation arising from Eq. (3.19); the minus sign in the left-hand side is thus seen to be a consequence of Eq. (4.11).

To summarize, we have proved that, in spite of the analogous structure of the corresponding Hamiltonians [Eqs. (1.8) and (1.10)], the C-T transformation does not lead to the extreme relativistic limit of the Klein-Gordon equation. This failure of the analogy will be now traced down to the appearance of the operator τ_2 in Eq. (1.10), instead of the operator τ_1 which would be expected from Eq. (1.8). This we do in Sec. 5.

5. *Nonunitary Transformations*

Let us consider again Eqs. (1.8) and (1.10). The operators $\mathbf{J}$ appearing in the former can be written in the form

$$\mathbf{J} = \tfrac{1}{2}\boldsymbol{\Sigma} \tag{5.1}$$

where the operators

$$\Sigma_1 \equiv (\boldsymbol{\alpha}\cdot\mathbf{p})/|\mathbf{p}|, \qquad \Sigma_2 \equiv -i\beta(\boldsymbol{\alpha}\cdot\mathbf{p})/|\mathbf{p}|, \qquad \Sigma_3 \equiv \beta \tag{5.2}$$

satisfy a Pauli algebra,

$$\Sigma_j\Sigma_k = i\epsilon_{jkl}\Sigma_l + \delta_{jk}$$

as can be readily verified, thus making the $\mathbf{J}$ operators the *exact analogs* of the $\boldsymbol{\tau}$ operators of Eq. (1.10).

Further, we remark that Dirac's original procedure to obtain his Hamiltonian is based on the fact (in our language) that the anticommutator

$$\{J_1, J_3\} = 0 \tag{5.3}$$

which implies that

$$(H_D)^2 = |\mathbf{p}|^2 + m^2 = E^2 \tag{5.4}$$

The corresponding Klein–Gordon analogs are the equations

$$\{\tau_2, \tau_3\} = 0 \tag{5.5}$$

and

$$(H_{KG})^2 = P^2 + M^2 = E^2 \tag{5.6}$$

Therefore, in the above sense P and M are the analogs in the Klein–Gordon theory of $|\mathbf{p}|$ and m, respectively, in the Dirac theory.

Let us now consider the Dirac equation written in its manifestly covariant form,

$$(\gamma^\mu p_\mu - m)\psi(\mathbf{p}) = 0$$

which in our notation is

$$(2i|\mathbf{p}|J_2 - 2p_0 J_3 + m)\psi(\mathbf{p}) = 0 \tag{5.7}$$

As Eq. (5.7) is obtained from the corresponding Hamiltonian equation by multiplying the latter throughout by the operator $2J_3(=\beta)$, we obtain the Klein–Gordon analog of Eq. (5.7) by multiplying the Hamiltonian Eq. (1.2) by the operator $2\tau_3$, which yields the result

$$(-2iP\tau_1 - 2p_0\tau_3 + M)\phi(\mathbf{p}) = 0 \tag{5.8}$$

and shows that in this case the correspondence $J_2 \to -\tau_1$ arises together with the already known $J_3 \to \tau_3$.

Now, it is clear that, given an angular momentum algebra (**J**, say), the algebra obtained through the relabeling

$$1 \to 2, \qquad 2 \to -1, \qquad 3 \to 3 \tag{5.9}$$

is also an angular momentum algebra; this explains the *necessity* of the minus sign appearing in Eqs. (4.6) and (4.10), as the rotation operator (3.10) contains the operator τ_1.

After these considerations we now return to the problem of finding a transformation suitable for the description of the extreme-relativistic limit of the Klein–Gordon equation, which, by comparison with Eqs. (3.2) and (3.7), we can expect to obtain by eliminating the operator τ_3 in Eq. (5.8). Before doing this, however, we briefly recall the nonunitary transformations associated with Eq. (5.7).

The transformation connecting the laboratory system ($\mathbf{p} = \mathbf{p}$) with the rest system ($\mathbf{p} = \mathbf{0}$) in the Dirac theory is conventionally (and somewhat improperly) called a Lorentz transformation. In our **J** formalism this is the *hyperbolic* rotation (i.e., the nonunitary transformation)

$$T = e^{iJ_1\theta} \tag{5.10}$$

which eliminates the operator J_2 in Eq. (5.7) with the choice of the angle

$$\tan\theta = i\,|\mathbf{p}|/p_0 \equiv \tan\theta_D^L \tag{5.11}$$

or, using the conventional notation $\theta = i\omega$,

$$\tanh \omega_D^L = |\mathbf{p}|/p_0 \tag{5.12}$$

On the other hand, if one chooses to eliminate the operator J_3 in Eq. (5.7) with the rotation (5.10), the corresponding angle is given by

$$\tan \theta = i p_0/|\mathbf{p}| \tag{5.13}$$

or

$$\tanh \omega = p_0/|\mathbf{p}| \tag{5.14}$$

which leads to a wave equation describing a "transcendent" tachyon (Saavedra, 1970), i.e., a tachyon with vanishing energy ($\mathbf{p}^2 = -m^2$, $v = \infty$).

The geometrical analogs of these transformations are obtained by remarking that they leave invariant the quadratic form

$$m^2 = p_0^2 - |\mathbf{p}|^2 \tag{5.15}$$

that is, they are the set of the (hyperbolic) rotations in the plane $(p_0, i|\mathbf{p}|)$, namely, the rotations

$$p_0' = p_0 \cos\theta + i|\mathbf{p}| \sin\theta \tag{5.16a}$$

$$i|\mathbf{p}'| = -p_0 \sin\theta + i|\mathbf{p}| \cos\theta \tag{5.16b}$$

The rotation for which $|\mathbf{p}'|$ vanishes (rest system) is given by

$$\tan\theta = i|\mathbf{p}|/p_0 = \tan\theta_D^L \tag{5.17}$$

that is, by the Lorentz angle. On the other hand, the rotation for which p_0' vanishes is given by

$$\tan\theta = ip_0/|\mathbf{p}| \tag{5.18}$$

which is Eq. (5.13) and is a tachyon transformation, as can be seen from Eq. (5.16b), which with the angle (5.18) yields the result

$$i|\mathbf{p}'| = -m \tag{5.19}$$

from which the relation $E = 0$ follows ("transcendent" tachyon).

Let us now consider the corresponding Klein-Gordon analog, i.e., the set of rotations [in the plane (p_0, iP)] leaving invariant the quadratic form

$$M^2 = {p_0}^2 - P^2 \tag{5.20}$$

The rotation for which P' vanishes, i.e., for which $|\mathbf{p}'| = 0$, is given by the angle

$$\tan\theta = iP/P_0 = -\mathbf{p}^2/2mp_0 \equiv \tan\theta_{KG}^{\text{``}L\text{''}} \tag{5.21}$$

which we can expect to be the (nonunitary) transformation taking the Klein-Gordon equation to its rest-system form—i.e., the Klein-Gordon "Lorentz" transformation.

To show that this is indeed the case we need only rotate Eq. (5.8) with the operator

$$T = e^{i\tau_2\theta} \tag{5.22}$$

which yields the result

$$\{\sigma_1 [(\mathbf{p}^2/2m) \cos\theta + p_0 \sin\theta] + \sigma_3 [(\mathbf{p}^2/2m) \sin\theta - p_0 \cos\theta] + M\}\tilde{\phi} = 0 \tag{5.23}$$

The operator σ_1 is therefore eliminated by the choice of angle

$$\tan\theta_{\mathrm{KG}}^{\text{``}L\text{''}} = -\mathbf{p}^2/2mp_0$$

which is Eq. (5.21). The resulting wave equation indeed has the rest system form,

$$\sigma_3\tilde{\phi} = \pm\tilde{\phi} \tag{5.24}$$

and is the F-W equation already considered [Eq. (3.14)]. However, the corresponding transformation [Eq. (5.22)] is nonunitary in this case, as the angle θ is real and we have

$$\sigma_3 T^+ \sigma_3 = T$$

instead of Eq. (2.10). The norm is consequently not invariant under this transformation, and we find by explicit calculation [using Eq. (2.13)],

$$\tilde{\phi}^+\sigma_3\tilde{\phi} = \frac{E^2 + m^2}{2mE} = \frac{M}{E} \tag{5.25}$$

Recalling the corresponding result (m/E) in the Dirac theory and its physical interpretation, we therefore conclude that the transformation (5.22) is "Lorentz" only in the sense that it is a nonunitary transformation taking Eq. (5.8) to its nonrelativistic (rest-system) form (this explains our use of quotation marks to refer to it).

We can now go back to Eq. (5.20) and consider the rotation given by

$$\tan\theta = ip_0/P = 2mp_0/\mathbf{p}^2 \tag{5.26}$$

for which we have $p_0' = 0$ and $iP' = -M$. We remark that the latter result is sufficient to obtain the former [Eq. (1.11)], and that [using Eqs. (1.9)] it leads to the solution $m = 0$, which is the condition to obtain the extreme-relativistic limit (Eq. (4.7)].

In the corresponding physical case we find that the angle given by Eq. (5.26) eliminates the operator σ_3 in Eq. (5.23), yielding the equation ($\hat{\phi} = T\phi$)

$$-\sigma_1\hat{\phi} = \hat{\phi} \tag{5.27}$$

for both signs of the energy; this equation is then suitable for the description of the ultrarelativistic limit (i.e., $|\mathbf{p}| \gg m$) of the Klein-Gordon equation. This assertion can be readily verified by directly taking the limit $|\mathbf{p}| \gg m$ in Eq. (5.8). We find first

$$(\sigma_3 + i\sigma_2)\hat{\hat{\phi}} = 0$$

which can indeed be given the form (5.27) upon multiplication by the operator σ_2.

The solutions of Eq. (5.27) have the form

$$\hat{\phi} = \begin{pmatrix} a \\ -a \end{pmatrix} \tag{5.28}$$

[a result which can also be obtained directly from Eq. (2.13) in the limit $|\mathbf{p}| \gg m$] with the norm

$$N(\hat{\phi}) = (\hat{\phi}, \sigma_3\hat{\phi}) = 0 \tag{5.29}$$

i.e., *the extreme-relativistic Klein–Gordon states have vanishing norm.* In the formalism we have used in this paper, on the other hand, the norm is understood as an electrical charge density; the interpretation of the result (5.29) is therefore that for a sufficiently fast particle the electrical charge disappears, that is, that the electrical charge is not conserved. We must conclude therefore that these states do not exist†.

Now, an extreme-relativistic Klein–Gordon particle is a massless zero-spin object, i.e., a Goldstone boson; our conclusion is therefore that *there are no Goldstone bosons in relativistic quantum mechanics.*

We can still go a step further using a "correspondence principle," relating quantum (second quantized) fields to quantum (first quantized) amplitudes. We recall that in quantum field theory Goldstone bosons must appear whenever we have a spontaneously broken symmetry, except for the case of gauge fields [Higgs' Theorem (Higgs, 1964)]. From this point of view we can hence conclude that the only physically meaningful fields are the gauge fields.

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† It is clear, as already discussed in the Introduction, that this argument applies only in the case $\rho(\mathbf{x}) \neq 0$; notice, however, that the *total charge*, $Q = \int d^3x\rho(\mathbf{x})$, may vanish. Further, it is worthwhile pointing out also that in a more complete theory, where other quantum numbers could be accounted for, a result like ours could be understood as a statement in the sense that a zero-mass (Klein–Gordon) particle must have zero electrical charge.

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